MATH 2040 Lecture 10 (Oct 13, 2016)

Reminder: Midterm 1

Next Tue (Oct 18) 7:30-9pm LSB LT 1

Topics: Lecture 1-10 (Textbook Ch. 1-4, 5.1, 5.2, 5.4)

Recall: (Cayley-Hamilton Theorem)

$T: V \to V$ linear (dim $< +\infty$), with char. poly. $f(t)$ $\Longrightarrow$ $f(T) = 0$ ← zero transformation

i.e. T "satisfies" the characteristic equation.

Lemma: $T: V \to V$, fix any $\vec{v} \in V$

T-cyclic subspace: $W = \text{span}\{\vec{v}, T\vec{v}, T^2\vec{v}, \ldots\} \subseteq V$

Assume $\dim W = k \leq n = \dim V$. Then,

(a) $\underbrace{\{\vec{v}, T\vec{v}, T^2\vec{v}, \ldots\ldots, T^{k-1}\vec{v}\}}_{k \text{ vectors in } W}$ is a basis for W

(b) $$T^k\vec{v} = -a_0\vec{v} - a_1T\vec{v} - \cdots - a_{k-1}T^{k-1}\vec{v} \qquad (*)$$

$\Rightarrow$ char. poly. of $T|_W = f_W(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k)$

Assume Lemma true.

Proof of C-H Thm:

Observe: (b) $\iff$ $f_W(T) := (-1)^k (a_0 I + a_1 T + \cdots + a_{k-1} T^{k-1} + T^k)$

(*) $\iff$ $\boxed{f_W(T)(\vec{v}) = \vec{0}}$

Claim: $f(T) = 0$, i.e. $f(T)(\vec{v}) = \vec{0}$ $\forall \vec{v} \in V$

Pf: Fix any $\vec{v} \in V$ ⇝ get W T-cyclic subsp. gen. by $\vec{v}$

arbitrary.

(*) $\Rightarrow$ $f_W(T)(\vec{v}) = \vec{0}$ $\Longrightarrow$ $f(T)(\vec{v}) = \vec{0}$

↑

$\because$ $f_W(T) \mid f(T)$ ▫

Proof of Lemma: (By induction on $k = \dim W$.)

(a) Let j be the smallest integer s.t. $(\vec{v} \neq \vec{0})$

$$\beta = \{\underbrace{\vec{v}, T\vec{v}, \ldots, T^{j-1}\vec{v}}_{j}\} \text{ lin. indep.}$$

Claim: $j = k$ $\Rightarrow$ (a)

Proof of Claim: $Z = \text{span}\,\beta \subseteq W$

Claim: $=$

Need: $Z \supseteq W$.

$\vec{v} \in$ ... ↖ smallest T-inv. subsp. contains $\vec{v}$

Strategy: Show Z is T-inv.

Check: pick any $\vec{w} \in Z$ (Q: $T\vec{w} \overset{?}{\in} Z$)

$$\vec{w} = a_0\vec{v} + a_1 T\vec{v} + \cdots + a_{j-1}T^{j-1}\vec{v}$$

$$T\vec{w} = \underbrace{a_0 T\vec{v} + a_1 T^2\vec{v} + \cdots + a_{j-1}T^{j-1}\vec{v}}_{\in Z} + \underbrace{a_{j-1}T^{j}\vec{v}}_{\in Z}$$

$\in Z$ ∵ choice of j

(b) Direct computation.

$f_W(t) =$ char. poly. of $T|_W$

(a) $\Rightarrow$ $\beta = \{\vec{v}, T\vec{v}, \dots, T^{k-1}\vec{v}\}$ basis for W

Compute $[T|_W]_\beta$

$$[T|_W]_\beta = \begin{pmatrix} 0 & & & 0 & -a_0 \\ 1 & \ddots & & & -a_1 \\ & \ddots & \ddots & 0 & \vdots \\ 0 & & & 1 & -a_{k-1} \end{pmatrix}$$

By induction (Ex.)

$$f_W(t) = \det([T|_W]_\beta - tI)$$

$$= (-1)^k (a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k)$$

E.g.: $\det\left[\begin{pmatrix} 0 & -a_0 \\ 1 & -a_1 \end{pmatrix} - t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$

$$= \det\begin{pmatrix} -t & -a_0 \\ 1 & -a_1 - t \end{pmatrix}$$

$$= t^2 + a_1 t + a_0$$

$$= (-1)^2 (a_0 + a_1 t + t^2) \#.$$

$\square$